

# Fractal space frames and metamaterials for high mechanical efficiency

R. S. FARR<sup>1</sup> and Y. MAO<sup>2</sup>

<sup>1</sup> Unilever R&D, Olivier van Noortlaan 120, AT3133, Vlaardingen, The Netherlands

<sup>2</sup> School of Physics and Astronomy, University of Nottingham, Nottingham, United Kingdom

PACS 46.32.+x – Static buckling and instability

PACS 46.25.Cc – Static elasticity: theoretical studies

PACS 46.70.Hg – Membranes, rods and strings

**Abstract.** – A solid slender beam of length  $L$ , made from a material of Young’s modulus  $Y$  and subject to a gentle compressive force  $F$ , requires a volume of material proportional to  $L^3 f^{1/2}$  [where  $f \equiv F/(YL^2) \ll 1$ ] in order to be stable against Euler buckling. By constructing a hierarchical space frame, we are able to systematically change the scaling of required material with  $f$  so that it is proportional to  $L^3 f^{(G+1)/(G+2)}$ , through changing the number of hierarchical levels  $G$  present in the structure. Based on simple choices for the geometry of the space frames, we provide expressions specifying in detail the optimal structures (in this class) for different values of the loading parameter  $f$ . These structures may then be used to create effective materials which are elastically isotropic and have the combination of low density and high crush strength. Such a material could be used to make light-weight components of arbitrary shape.

**Introduction.** – It has recently been shown that fractal [1] design principles can be used in structures to improve their mechanical efficiency under conditions of gentle compressive loading [2, 3]. The purpose of this paper is to introduce much simpler structures which display the same phenomenon, are more practical to manufacture, and which can then be used as components to construct an effective composite material (metamaterial). This material will, on long length scales, be elastically isotropic, and have a very low density but a comparatively high compressive strength or ‘crush pressure’ [4]. It could in principle then be used to manufacture light-weight components of any 3d form, in the same way that solid foams are used currently [5]. Being composed of connected, straight beams, this structure should be amenable to modern fabrication techniques [6–8].

To begin the argument, we consider fractal trusses or space-frames [4]. We wish to design a structure made from a volume  $V$  of material of Young’s modulus  $Y$ , which is required to support a compressive force  $F$  applied at freely hinged end-points, separated by a distance  $L$ . It is then useful to define a non-dimensionalized force parameter

$$f \equiv F/(YL^2) \ll 1 \quad (1)$$

and non-dimensionalized volume

$$v \equiv V/L^3 \ll 1. \quad (2)$$

The aim is to find a structure which, for small values of  $f$ , minimizes  $v$ . In particular, the scaling of achievable values of  $v$  with  $f$ , as  $f \rightarrow 0$  is of interest.

For a solid cylindrical column, the limitation is Euler buckling [9], which leads to  $v = 2\pi^{-1/2} f^{1/2}$  (in contrast to the much higher efficiencies of  $v \propto f$  possible for structures under tension) [4, 10].

**A space-frame.** – Instead of a single solid cylindrical column, let us consider a space-frame. For ease of analysis (rather than because this is the global optimum) the structure is composed of beams of equal unstressed length  $L_0$ , which are freely hinged to each other at their ends.

In cartesian co-ordinates, the end points of the structure lie at the origin and the point  $(0, 0, L)$ . The geometry of the space frame consists conceptually of a regular tetrahedron at each of its ends, and between these a stack of  $n \geq 0$  regular octahedra, as shown stereographically in fig. 1(a) and (b) for the cases  $n = 0$  and 3 respectively. Therefore the length of each component beam is

$$L_0 \equiv \frac{1}{2} \sqrt{6} L/(n+2). \quad (3)$$

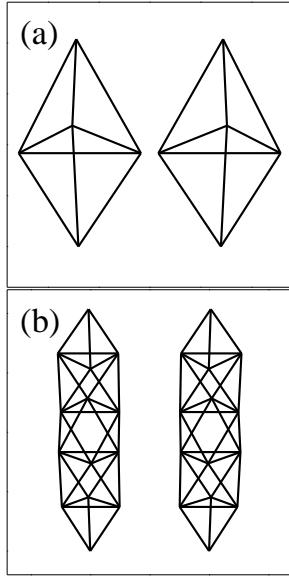


Fig. 1: Stereograms of ‘generation 1’ space-frames, with (a)  $n_{1,1} = 0$  and (b)  $n_{1,1} = 3$ . To view these figures, hold the page about 30cm away, and look through the page, until the two images merge.

A consequence of the rigid polyhedron theorem [11] is that space frames which are convex triangular polyhedra composed of rigid beams are themselves rigid, and therefore this design based on tetrahedra and octahedra ensures that there are no soft modes for the structure: any flexibility must involve stretching or compressing the constituent beams.

When a small compressive load  $F$  is applied to the ends of the space frame, some of the component beams will be under tension, and others under compression. In particular, the beams which are parallel to the  $x - y$  plane are all under tension, and the rest are under compression. The six beams connected directly to the ends are all under a compressional force

$$F_{\text{com}} = F/\sqrt{6}, \quad (4)$$

while the other beams under compression support only half this load apiece.

For the beams under tension, if  $n = 0$ , then all three are subject to a force of  $2F/(3\sqrt{6})$ . On the other hand, if  $n \geq 1$ , the six tension beams closest to the end points are subject to a force  $F/(2\sqrt{6})$ , and if  $n \geq 2$ , then all the remaining tension beams are subject to a force  $F/(3\sqrt{6})$ .

Each of the beams will behave as a linear spring under stretching or compressional forces small enough to avoid Euler buckling of these individual beams. We now suppose that all of the beams have the same effective spring constant (force per unit extension), given by  $k$ . Once more this choice is made for simplicity of analysis: we note as an aside that more detailed calculation shows that a gain in efficiency by a numerical factor of order unity is possible by choosing some beams to have different spring con-

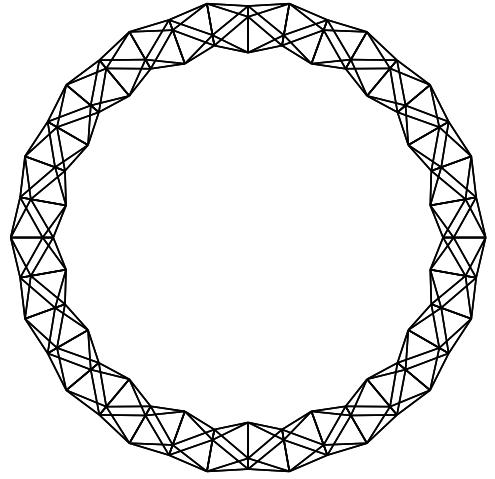


Fig. 2: Space frame consisting of 36 octahedra composed of equal beams, connected to form a circle, and allowed to relax. The elastic energies for space frames of this type (and varying numbers of octahedra) are used to estimate the bending stiffness of the space frame, treated as an equivalent slender beam.

stants; however, the functional dependence of  $v$  of  $f$  is not affected.

With this assumption of equal spring constants  $k$ , calculation shows that the spring constant of the entire space frame under compression is given by

$$K = \frac{36k}{11n + 43}, \quad (5)$$

provided  $n \geq 3$ .

Next, provided  $n$  is large, the entire space frame will resemble a long, slender beam, with a bending stiffness  $YI$ , where  $I$  is the second moment of the area about the neutral axis for the equivalent slender beam [12]. Because of the three-fold symmetry of the space frame on rotation about the  $z$ -axis, and the possible symmetries of the second moment calculation, we would expect the bending stiffness to not depend on the direction of bending.

Finally, we would expect that as  $n \rightarrow \infty$ ,

$$YI \rightarrow BL_0^3k, \quad (6)$$

for some numerical constant  $B$ . This constant is difficult to calculate analytically. However, by simulating long space frames which have their ends joined to form a circle, and which are allowed to relax (fig. 2), the relaxed energy for different numbers of constituent octahedra can be calculated. By choosing the number of octahedra up to 128 and extrapolating to higher values, we find

$$B = 0.245 \pm 0.001. \quad (7)$$

During Euler buckling of a freely hinged beam, the curvature at the end points will vanish [12] (this follows directly from the freely hinged condition). Therefore using

eq. (6) in the expression for the maximum force  $F_{\text{buc}}$  which a slender beam can sustain before buckling [12]:

$$F_{\text{buc}} = \frac{\pi^2 Y I}{L^2} \quad (8)$$

will provide a good approximation to the failure criterion for the whole space frame under a global buckling instability, even when  $n$  is not large.

Each component beam is also vulnerable to buckling (in fact the component beams which are joined to the end points are most vulnerable), and therefore the force  $F$  which the structure can withstand is subject to two constraints: the individual beams must not buckle, and the space frame itself should not undergo a global buckling instability. In order to be optimally efficient, the structure should be on the verge of both instabilities.

Once we have chosen  $L$ , the space frames we have just described are specified by two parameters: the number of octahedra  $n$ , and the radius  $r$  of the circular cross-section of the component beams. This last is related to the spring constant of the beams through

$$r = \left( \frac{k L_0}{\pi Y} \right)^{1/2} \quad (9)$$

[with  $L_0$  coming from eq. (3)].

To design an optimal structure, one should therefore specify  $L$ ,  $Y$  and  $F$  (or, more elegantly, the parameter  $f$ ), and then choose the values of  $n$  and  $r$  which minimize  $v$ . This approach (especially for the hierarchical structures described below where there are different  $n$ 's), leads to equations which are difficult to solve analytically. Instead, we construct optimal structures through a simpler algorithm, which is possible because of the minimal coupling between hierarchical levels in the structure. This generalizes easily to more complex structures, and is implemented as follows:

First, we parameterize the optimum space frames through a new dimensionless parameter

$$f_0 \equiv (F/\sqrt{6})/(Y L_0^2), \quad (10)$$

which is the  $f$  parameter calculated for one of the single constituent beams which is most vulnerable to buckling.

The condition that this beam be at the Euler buckling limit then allows us to calculate  $r$ , giving

$$r = L_0 \left( \frac{4 f_0}{\pi^3} \right)^{1/4}. \quad (11)$$

Second, the condition that the entire structure not buckle globally allows us to determine  $n$ , by using eqs. (3), (6) and (8) - (11):

$$\begin{aligned} n &= -2 + \left[ \left( \frac{3\pi^3}{2} \right)^{1/4} B^{1/2} f_0^{-1/4} \right] \\ &\approx -2 + [1.29 f_0^{-1/4}], \end{aligned} \quad (12)$$

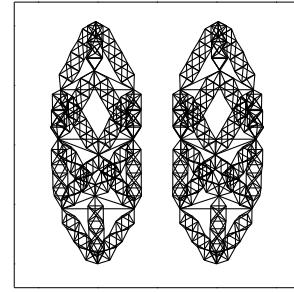


Fig. 3: Generation 2 space frame, with  $n_{2,1} = 4$  and  $n_{2,2} = 2$ .

where  $\lfloor \cdot \rfloor$  is the floor function.

The third step is to calculate  $f$  and  $v$  in terms of  $f_0$ , which follow directly from eqs. (4), (3) and (12) as

$$f \approx 3.67 f_0 \left( \lfloor 1.29 f_0^{-1/4} \rfloor \right)^{-2}, \quad (13)$$

$$v \approx 18.7 \left( -1 + \lfloor 1.29 f_0^{-1/4} \rfloor \right) \left( \lfloor 1.29 f_0^{-1/4} \rfloor \right)^{-5}. \quad (14)$$

For small  $f$ , we therefore have

$$v \propto f^{2/3}, \quad (15)$$

which represents a qualitative increase of efficiency over a solid cylindrical column, for which  $v \propto f^{1/2}$ .

**Hierarchical space frames.** – In the last section, we described a simple space frame, consisting of solid cylindrical beams. We now re-name this a ‘generation  $G = 1$ ’ structure, which for fixed  $L$  can be specified by the radius  $r$  of the cross section of the constituent beams, and the number of constituent octahedra, which we now re-name  $n_{1,1}$ . In this notation, the first index refers to the generation number  $G$ , and the second to the depth in the hierarchical structure (see below).

To define a generation  $G = 2$  structure, we replace all the compressional beams in a generation 1 structure, by (scaled) generation 1 space frames, which are all identical to each other. The number of octahedra present in the two hierarchical levels of the structure may however be different; with  $n_{2,1}$  being the number of octahedra in the smallest component space frames, and  $n_{2,2}$  being the number of large octahedra (composed in part of composite beams) in the entire structure. However, for simplicity of manufacture, the tension members are allowed to remain as simple solid cylinders, since no gain in efficiency is derived from making them more complex. An example is shown in fig. 3, with  $n_{2,1} = 4$  and  $n_{2,2} = 2$ .

This process may be repeated, in each iteration replacing the smallest compressional beams by identical (scaled) generation 1 space frames, and so obtaining structures with higher and higher generation number  $G$ . fig. 4 shows a generation 3 space frame, with  $n_{3,1} = 4$ ,  $n_{3,2} = 3$  and  $n_{3,3} = 2$ .

In the final generation  $G$  structure, all the compressional beams are solid cylinders of equal length  $L_{G,0}$ , and have

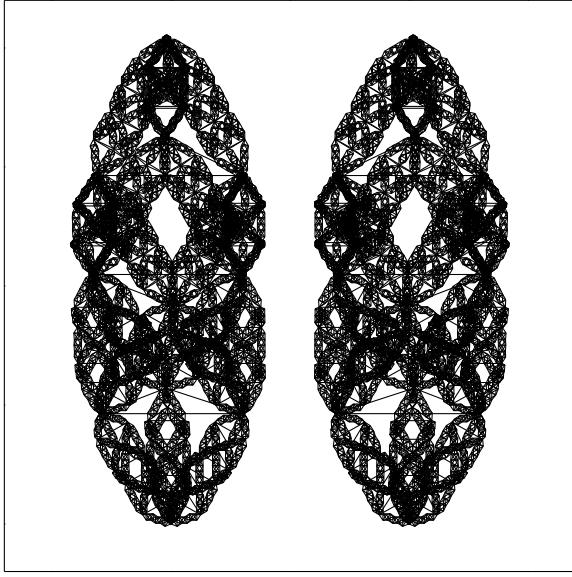


Fig. 4: Generation 3 space frame, with  $n_{3,1} = 4$ ,  $n_{3,2} = 3$  and  $n_{3,3} = 2$ .

equal radii  $r$ . These are the smallest beams in the structure. They have a bending stiffness  $YI_{G,0} = \pi Y r^4/4$ , a spring constant before buckling of  $k_{G,0} = \pi Y r^2/L_{G,0}$  and they support a maximum compressional force of  $F_{G,0}$ .

The smallest beams make up small space frames which have a geometry similar to generation 1 structures. Each of these space frames are of length  $L_{G,1}$ , and contain  $n_{G,1}$  octahedra. They have a bending stiffness  $YI_{G,1}$ , spring constant  $k_{G,1}$  and support a maximum compressive load of  $F_{G,1}$ .

Higher levels in the hierarchical structure are defined in the same way, so that eventually

$$L \equiv L_{G,G} \quad (16)$$

$$F \equiv F_{G,G}. \quad (17)$$

From eqs. (3), (6) and (5), we can write down recurrence relations for all these quantities, valid for  $1 \leq i \leq G$ :

$$L_{G,i} = \left(\frac{2}{3}\right)^{1/2} (n_{G,i} + 2)L_{G,i-1} \quad (18)$$

$$YI_{G,i} = BL_{G,i-1}^3 k_{G,i-1} \quad (19)$$

$$k_{G,i} = \frac{36k_{G,i-1}}{11n_{G,i} + 43} \quad (20)$$

$$F_{G,i} = \sqrt{6}F_{G,i-1}. \quad (21)$$

There will also be tensional beams in the structure of various lengths, which have been ‘left over’ - i.e. not replaced by composite beams - in the process of constructing the structure. The radii  $t_{G,i}$  of these tensional beams are chosen (for simplicity of analysis) to give them the same spring constant as the compressional (usually composite) beams at the same level in the hierarchical structure. Thus

the smallest tensional beams are identical to the compressional beams

$$t_{G,0} \equiv r, \quad (22)$$

and in general the tensional beams at level  $i \in (0, G-1)$  have length  $L_{G,i}$  and radius

$$t_{G,i} = \left(\frac{L_{G,i}k_{G,i}}{\pi Y}\right)^{1/2}. \quad (23)$$

Note: there is no tensional beam at level  $G$ , since this is the entire structure, which is under compression.

**Optimization.** – As with the simple space frames above, we parameterize optimized space frames through

$$f_0 \equiv F_{G,0}/(YL_{G,0}^2). \quad (24)$$

The first stage is to find the optimal choices for  $n_{G,i}$  at each level in the structure. To do this, let us take  $L_{G,0}$  as fixed (we will determine it in terms of  $L$  only at the end of the calculation). We can then determine the radius of the smallest compression beams though imposing the condition that they be on the verge of Euler buckling:

$$r = L_{G,0} \left(\frac{4f_0}{\pi^3}\right)^{1/4}, \quad (25)$$

so that

$$k_{G,0} = 2\pi^{-1/2}L_{G,0}Yf_0^{1/2}. \quad (26)$$

The condition for buckling to not occur at level  $i \in (1, G)$  in the structure is

$$F_{G,i} \leq \frac{\pi^2 Y I_{G,i}}{L_{G,i}^2}, \quad (27)$$

which from eq. (18), (19) and (20) leads for  $i = 1$  to

$$n_{G,1} \approx -2 + \lfloor 1.29f_0^{-1/4} \rfloor, \quad (28)$$

and for  $2 \leq i \leq G$ , we find from eqs. (18-21), (26) and (27) that

$$n_{G,i} = -2 + \left[ A f_0^{-1/4} (2\sqrt{3})^i \prod_{j=1}^{i-1} \left( \frac{n_{G,j} + 2}{11n_{G,j} + 43} \right)^{1/2} \right], \quad (29)$$

where

$$A = \frac{B^{1/2} \pi^{3/4}}{2(6)^{1/4}} \approx 0.373. \quad (30)$$

eqs. (28) and (29) allow us to calculate all the values for numbers of octahedra at different levels in the structure. A simple calculation (including the volume of the tension beams) then leads, for  $G \geq 2$  to

$$f = \left(\frac{27}{2}\right)^{G/2} f_0 \prod_{j=1}^G (n_{G,j} + 2)^{-2} \quad (31)$$

Table 1: Example calculation for the mass  $M$  if a structure required to support  $F = 10\text{kN}$  over a distance of  $L = 200\text{m}$  when the structure is made from a material similar to steel, with  $Y = 210\text{GPa}$  and density  $\rho = 8000\text{kgm}^{-3}$ . This corresponds to  $f = 1.2 \times 10^{-12}$ .

$G$	$f_0$	$n_{G,1}$	$n_{G,2}$	$n_{G,3}$	$M$
0	$1.2 \times 10^{-12}$				79 tonnes
1	$6.63 \times 10^{-9}$	140			2920kg
2	$4.92 \times 10^{-7}$	46	47		1790kg
3	$6.36 \times 10^{-6}$	23	23	24	2180kg

$$v = \frac{1}{\sqrt{\pi}} \left( \frac{243}{2} \right)^{G/2} \left[ \prod_{j=1}^G \frac{(n_{G,j} + 1)}{(n_{G,j} + 2)^3} \right] f_0^{1/2} \times \left( 3 + \sum_{q=1}^{G-1} \left\{ \prod_{j=1}^q \left[ \frac{2^{2q}(n_{G,j} + 2)^2}{(11n_{G,j} + 43)(n_{G,j} + 1)} \right] \right\} \right). \quad (32)$$

To illustrate these calculations, consider a space frame of length  $L = 200\text{m}$  which is required to support a force of  $F = 10\text{kN}$ , and which is made from a model material, similar to steel, with  $Y = 210\text{GPa}$  and a density of  $8000\text{kgm}^{-3}$ , so that  $f = 1.2 \times 10^{-12}$ .

A cable supporting this force under tension would require a mass of 8kg (assuming a yield stress for the material of 200MPa, and neglecting the mass of couplings at the ends). The masses ( $M$ ) of “steel” required for various structures described in this paper are shown in table 1.

More generally, eqs. (31) and (32) show that for fixed  $G$ , then as  $f \rightarrow 0$ ,

$$v \propto f^{(G+1)/(G+2)}. \quad (33)$$

Furthermore, for any particular  $f$ , there will be an optimum generation number  $G_{\text{opt}}$ , which gives the most efficient (smallest  $v$ ) structures in this class. table 2 shows the approximate ranges of  $f$  which correspond to different values of  $G_{\text{opt}}$ .

Finally, we note that there is still considerable opportunity to optimize the space frames presented here; the design in this paper has been motivated by a search for the scaling behaviour in the limit of small  $f$ . If one wished to reduce the pre-factor, there are opportunities through allowing different beams to have different thicknesses in the structure (currently the structure has weak points near the ends), through choosing a different basic space frame geometry, and potentially also through moving from freely hinged to rigid boundary conditions at some of the joints. Such improvements will also alter the crossover values in table 2.

**An elastic metamaterial.** – For application to an elastically isotropic composite material, we note that for example, a face centred cubic lattice built from freely hinged beams between nearest and second neighbours [13] has this property of isotropy. If we replace all these beams with the hierarchical structures described in this paper,

Table 2: The approximate ranges of  $f$  and  $f_0$ , for which different generation numbers  $G_{\text{opt}}$  give the global optimum space frames of the type described in this paper.

$G_{\text{opt}}$	range of $f$	range of $f_0$
0	$10^{-4} < f$	$10^{-4} < f_0$
1	$10^{-9} < f < 10^{-4}$	$10^{-6} < f_0 < 10^{-3}$
2	$10^{-14} < f < 10^{-9}$	$10^{-7} < f_0 < 10^{-5}$
3	$10^{-62} < f < 10^{-14}$	$10^{-25} < f_0 < 10^{-6}$

then the resulting effective material will use a volume fraction  $\phi$  of real material, where  $\phi$  is of order  $v$ . The crush pressure of the composite material will, from eq. (33), scale with volume fraction according to

$$p_c \sim Y\phi^{(G+2)/(G+1)}. \quad (34)$$

For small enough  $\phi$ , the value of  $p_c$  according to eq. (34) is much larger than a material built from a lattice of simple beams. We note that cancellous bone (which is fractal [14]) also has an unusual scaling of (anisotropic) compressive strength with density [5].

**Other detailed considerations.** – We note that in the initial specification of the space frame, we have required all the component beams to be freely hinged to one another. Potentially, these hinges require a certain amount of additional material. However, the extra material to make couplings for the ends of a beam is of order the cube of the beam diameter [10]. Therefore, including all the small beams, we expect this consideration to increase the amount of material in the overall structure by a factor of  $(1+\epsilon)$ , where  $\epsilon = \text{ord}(r/L_{G,0}) = \text{ord}(f_0^{1/4}) \ll 1$ , provided  $f_0 \ll 1$ .

The main reason for imposing freely hinged joints is that without this degree of freedom, the space frame would deform under loads, leading to the beams being no longer straight, even before the nominal buckling load. Such deformations could potentially weaken the structure. However even with freely hinged joints the structure will deform before failure, and although all the beams remain straight, the directions of force transfer will be slightly altered, which could disturb the calculations presented above. We would therefore like the elastic deformation of the structure at the time of failure to be small. Once again, we can check this directly: the compressional strain of the smallest beams at failure is [from eq. (25)] proportional to  $f_0^{1/2} \ll 1$ , provided  $f_0 \ll 1$ .

Lastly, we make an observation on the geometry of the structure: if we have a space frame as described in the sections above, with large  $G$ , and all the numbers  $n_{G,i}$  equal, then this structure will be a fractal over a suitable range of length scales. Furthermore, the Hausdorff dimension [15] will be a (decreasing) function of  $n_{G,i}$ . However, we see from eq. (29) that for optimized structures,  $n_{G,i}$  depends on the depth  $i$  in the hierarchy: in particular, as  $f \rightarrow 0$ , we find  $n_{G,i} \propto (12/11)^{(i-1)/2}$ . This leads to a change of Hausdorff dimension with length scale.

**Conclusions.** – We have presented an hierarchical design for space frames which allows us to systematically change the scaling of material needed versus compressive force under conditions of gentle loading. These can then be used to construct a light, strong metamaterial.

The resulting expressions for space frame efficiency are similar to those of curved shell struts [3], but the latter are more efficient than the designs presented in this paper. Considerable scope for further optimization of the prefactors however remains.

Both these approaches are an attempt to solve an optimization problem which is rather simply stated: ‘given the loading condition specified by  $f$ , what geometry of an elastic material minimizes  $v$ ?’. On the basis of recent work in this area, the authors conjecture that for small enough  $f$ , the answer will consist of some kind of hierarchical structure. However, mathematical tools for addressing optimization problems of this kind are notable mainly for their absence, and even recent numerical approaches [16,17] seem not to be powerful enough to tackle the problem directly.

\*\*\*

The authors wish to thank Dr Joel Segal of Nottingham University and Andrew Matthews of Devon County Council for useful discussions.

## REFERENCES

- [1] MANDELBROT B. B., *The Fractal Geometry of Nature* (W. H. Freeman & Co., New York) 1983
- [2] FARR R. S., *Phys. Rev. E*, **76** (2007) 046601
- [3] FARR R. S., *Phys. Rev. E*, **76** (2007) 056608
- [4] GORDON J. E., *Structures* (Penguin Books Ltd.) 1986
- [5] GIBSON L. J. and ASHBY M. F., *Cellular Solids, structure and properties - 2<sup>nd</sup> Ed<sup>n</sup>.* (Cambridge University Press) 1997
- [6] CHEN W. F. and RICHARD-LIEW J. Y., *The Civil Engineers' Handbook (Second Edition)* (CRC Press LLC) 2003
- [7] DAVIDSON B. and OWENS G. W., *Steel Designers' Manual* (WileyBlackwell) 1994
- [8] WRIGHT P. K., *21<sup>st</sup> Century Manufacturing* (Prentice Hall) 2000
- [9] VAGLIENTE V. N. and KRAWINKLER H., *J. Eng. Mech.-ASC E*, **113** (1987) 186
- [10] COX H. L., *The design of structures of least weight* (Pergamon Press, Oxford) 1965
- [11] CAUCHY A. L., *XVIIe cahier*, **IX** (1813) 87-89
- [12] TIMOSHENKO S. P. and GERE J. M., *Theory of Elastic Stability* (McGraw Hill) 1986
- [13] DONZE F. and MAGNIER S. A., *Geophysical Journal International*, **122** (1995) 790
- [14] HUISKES R., *Nature*, **405** (2000) 704
- [15] HAUSDORFF F., *Matt. Ann.*, **79** (1919) 157
- [16] BENDSOE M. P. and SIGMUND O., *Archive of Applied Mechanics*, **69** (1999) 635
- [17] ESCHENAUER H. A. and OLHOFF N., *Appl. Mech. Rev.*, **54** (2001) 331